VARIATIONS ON AUTOMATIC CONTINUITY

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ABSTRACT. If G is a group with the automatic continuity property, when is it the case that the group $G^{\mathbb{N}}$ also has the automatic continuity property?

CONTENTS

We are interested in recovering the topology on a Polish group from its algebraic structure, by means of the *automatic continuity property*. A separable topological group G has the automatic continuity property if every group homomorphism from G to any separable group is continuous.

However strong this may seem, many Polish groups satisfy the automatic continuity property; we refer the reader to Rosendal's very good survey [R] for more details. We are interested in finding more of those, by looking at infinite powers of Polish groups that satisfy the automatic continuity property. Such powers do not always have the automatic continuity property, even (if not especially) in the simplest of cases. Yet, they do when the Polish groups in question have *ample generics*, a very strong topological property. We prove that they also do with a weaker requirement: in the very particular framework, introduced by Sabok ([S1]) and Malicki ([M1]), where automatic continuity of the automorphism group is witnessed by specific combinatorial properties of the structure.

Moreover, in the course of a discussion on this question with François Le Maître, we discovered connected Polish groups with ample generics, answering a question of Kechris and Rosendal (see theorem 48).

1. Automatic continuity

Definition 1. Let G be a separable topological group. We say that the group G satisfies the **automatic continuity property** if every group homomorphism from G to any separable group is continuous.

Note that the separability assumption on the range group is necessary: without it, we can always endow the group G with the discrete topology and the identity map of G may fail to be continuous.

Proposition 2. A finite product of groups that all satisfy the automatic continuity property also satisfies the automatic continuity property.

Proof. Let G_1, ..., G_n be topological groups that satisfy the automatic continuity property. Let H be a separable group and let $\varphi : G_1 \times ... \times G_n \to H$ be a group homomorphism. For each i, consider the group homomorphism $\varphi_i : G_i \to H$ defined by $\varphi_i(g_i) = \varphi(1, ..., 1, g_i, 1, ..., 1)$. Since each G_i satisfies the automatic continuity property, all of the homomorphisms φ_i are continuous. Since there are finitely many groups in the product, we can write φ as $g \mapsto \varphi_1(g)...\varphi_n(g)$ so φ is continuous. ☐

However, the automatic continuity property does not go to infinite products in general. The following will be our companion (non-)example throughout this note.

Example 3. The group $\mathbb{Z}/2\mathbb{Z}$ is discrete. Thus, it satisfies the automatic continuity property. However, the group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ does not have the automatic continuity property. Indeed, let $\mathcal{U}$ be any non-principal ultrafilter on $\mathbb{N}$. It corresponds to a subgroup $H_{\mathcal{U}}$ of $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ of index 2, which is thus normal, and since $\mathcal{U}$ is non-principal, the subgroup $H_{\mathcal{U}}$ is not open. But then the group homomorphism from $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ into $\mathbb{Z}/2\mathbb{Z}$ of kernel $H_{\mathcal{U}}$ cannot be continuous.

The general question we would like to address is the following.

Question 4. If G is a group with the automatic continuity property, when is it the case that the group $G^{\mathbb{N}}$ also has the automatic continuity property?

In this chapter, we only touch upon this question. We answer it for Polish groups, seen as automorphism groups, in the particular case when automatic continuity results from combinatorial properties of the structure.

1.1. The Steinhaus property.

Rosendal and Solecki introduced in [RS] a very useful (and essentially the only) tool to prove the automatic continuity property.

Definition 5. Let G be a topological group. We say that the group G is **Steinhaus** if for every countably syndetic[1] subset W of G, there exists an integer k such that W^k contains an open neighborhood of the identity.

Theorem 6. (Rosendal-Solecki, [RS, proposition 2]) Let G be a separable topological group. If G is Steinhaus, then G satisfies the automatic continuity property.

Example 7. Since the group $\mathbb{Z}/2\mathbb{Z}$ is discrete, it is Steinhaus. However, the group $(\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ is not (otherwise, it would have the automatic continuity property).

It is unclear whether a finite product of Steinhaus groups also is Steinhaus. But since the Steinhaus property was introduced with the sole aim of proving the automatic continuity property, it does not matter too much in view of proposition 2.

We would like to argue that the group $\mathbb{Z}/2\mathbb{Z}$ has the automatic continuity property for the wrong reason (a trivial reason, discreteness).

[1] A subset W of G is said to be **countably syndetic** if there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of elements of G such that $G = \bigcup_{n\in\mathbb{N}} g_n W$.

1.2. **Better reasons than discreteness to be Steinhaus: ample generics.** Kechris and Rosendal introduced in [KR] the property of having *ample generics* for a topological group, and they proved that if a topological group has ample generics, then it satisfies the automatic continuity property ([KR, theorem 6.24]).

Definition 8. A subset of a Polish space X is called **comeager** if it contains a countable intersection of dense open subsets of X.

Definition 9. Let G be a topological group. We say that G has **ample generics** if for every positive integer n, the diagonal conjugacy action of G on G^n, which is given by

$$g \cdot (g_1, ..., g_n) = (g^{-1} g_1 g, ..., g^{-1} g_n g),$$

admits a comeager orbit.

Many closed subgroups of S_∞ have ample generics: the automorphism groups of the random graph, of the rational Urysohn space, of the infinitely splitting regular rooted tree. More generally, if a Fraïssé class satisfies two combinatorial properties, namely the extension property and the free amalgamation property (see sections 3 and 4 for a definition), then the automorphism group of its Fraïssé limit has ample generics (see [M2, theorem 4.5]).

However, bigger groups often fail to have ample generics. For instance, in the groups $\mathrm{Iso}(\mathbb{U})$, $\mathrm{Aut}(\mu)$ and $\mathcal{U}(\ell^2)$, every conjugacy class is meager. Actually, Kechris and Rosendal asked in [KR, question 6.13 (1)] whether there exists Polish groups with ample generics that are not subgroups of S_∞. With Le Maître ([KLM]), we exhibited an example of such a group (see theorem 48).

For our purposes, ample generics come out as particularly powerful, for they do carry to infinite powers.

Proposition 10. Let G be a topological group with ample generics. Then $G^{\mathbb{N}}$ also has ample generics.

Proof. Let n be an integer. We can naturally identify the group $(G^{\mathbb{N}})^n$ with $(G^n)^{\mathbb{N}}$. Since G has ample generics, there exists a tuple $\bar{\varphi}$ in G^n whose orbit is comeager in G^n. Now consider the constant sequence $\bar{f} = (\bar{\varphi})_{i\in\mathbb{N}}$ in $(G^n)^{\mathbb{N}}$. We prove that the orbit of $\bar{f}$ under the diagonal action of $G^{\mathbb{N}}$ is comeager in $(G^n)^{\mathbb{N}}$.

The orbit of $\bar{\varphi}$ contains a dense G_δ subset : $G \cdot \bar{\varphi} \supseteq \bigcap_{k\in\mathbb{N}} U_k$, where each U_k is a dense open subset of G^n. Then we have

$$\begin{aligned} G^{\mathbb{N}} \cdot \bar{f} &= \{(\bar{g}_i)_{i\in\mathbb{N}} \in (G^n)^{\mathbb{N}} : \text{ for all } i \text{ in } \mathbb{N}, \bar{g}_i \in G \cdot \bar{\varphi}\} \\ &= \bigcap_{i\in\mathbb{N}} \{(\bar{g}_i)_{i\in\mathbb{N}} \in (G^n)^{\mathbb{N}} : \bar{g}_i \in G \cdot \bar{\varphi}\} \\ &\supseteq \bigcap_{i\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} \{(\bar{g}_i)_{i\in\mathbb{N}} \bar{g}_i \in U_k\} \\ &= \bigcap_{i\in\mathbb{N}} \bigcap_{k\in\mathbb{N}} A_{i,k}. \end{aligned}$$

Since $(G^n)^{\mathbb{N}}$ is endowed with the product topology, each of the $A_{i,k}$'s is open and dense in $(G^n)^{\mathbb{N}}$, hence the orbit of $\bar{f}$ is comeager and $G^{\mathbb{N}}$ has ample generics. □

We will see in section 6 a generalization of this theorem to the group of G-valued random variables.

1.3. **Mimicking ample generics.** As mentioned before, ample generics fail for quite a number of big Polish groups. Yet, it is still possible to circumvent their absence: before they knew that the group Homeo($2^{\mathbb{N}}$) had ample generics[2], Rosendal and Solecki ([RS, theorem 13]) managed to proved the automatic continuity property for both Homeo($2^{\mathbb{N}}$) and Homeo($2^{\mathbb{N}}$)$^{\mathbb{N}}$. Drawing inspiration from their arguments, Sabok then introduced in [S1] a set of properties of *exactly* ultrahomogeneous metric structures that imply the Steinhaus property for their (big) automorphism groups. These conditions include the *existence property*, which is in some way similar to the free amalgamation property, and the extension property. Later, Malicki proposed in [M1] a slightly different set of properties, designed to imply not only the automatic continuity property but also several others consequences of ample generics (see [KR]). In the light of proposition 10, this set of properties that mimics ample generics is a reasonable condition to consider for our problem.

Their results are the following, with the different properties to be specified and discussed later on.

Theorem 11. (Sabok, Malicki) Let M be an exactly ultrahomogeneous metric structure. Assume that M has the extension property, the existence property and an isolation property. Then the automorphism group of M is Steinhaus and thus satisfies the automatic continuity property.

Remark 12. Again, the question arises of exact ultrahomogeneity and of a possible finitary characterization for it...

Corollary 13. The following groups have the automatic continuity property.

- Aut(μ) (Ben Yaacov-Berenstein-Melleray, [BBM, theorem 6.2]).
- $\mathcal{U}(\ell^2)$ (Tsankov, [T]).
- Iso($\mathbb{U}$) and Iso($\mathbb{U}_1$) (Sabok, [S1, section 8]).

We would like to investigate these properties and study how they behave with respect to products. In order to do that, given a metric structure and its automorphism group G, we need to exhibit a structure of which $G^{\mathbb{N}}$ is the automorphism group.

2. The juxtaposed structure

Let M be a metric structure of diameter smaller than 1 and let G be its automorphism group. The **juxtaposed structure** M^* of M consists of countably many copies of the structure M that do not interact with one another, together with a distinguished element $\star$ (that constitutes the zeroth "copy"). We endow the space $(\{\star\} \times \{0\}) \cup (M \times \mathbb{N} \setminus \{0\})$ with

- a unary predicate C_n for each copy $M \times \{n\}$,
- a unary predicate $C_\star$ for the element $(\star, 0)$,
- the metric defined by

$$d((a,i),(b,j)) = \begin{cases} d_M(x,y) & \text{if } i = j \neq 0 \\ 0 & \text{if } i = j = 0 \\ 1 & \text{if } i \neq j, \end{cases}$$

- a predicate P^* for each predicate P in M, defined by

$$P^*((a_1,i_1),...,(a_m,i_m)) = \begin{cases} P(a_1,...,a_m) & \text{if } i_1 = ... = i_m \neq 0 \\ 1 & \text{otherwise.} \end{cases}$$

[2] This was proved later by Kwiatkowska in [K2].

- a function F^* for each definable function $F : M^m \to M$ defined by

$$F^*((a_1, i_1), ..., (a_m, i_m)) = \begin{cases} (F(a_1, ..., a_m), i_1) & \text{if } i_1 = ... = i_m \neq 0 \\ (\star, 0) & \text{otherwise.} \end{cases}$$

Remark 14. The additional element $\star$ is designed to define functions. If the structure is relational, we can just take M^* to be the product space $M \times \mathbb{N}$ together with the appropriate predicates.

Since there is a predicate for each copy of M, automorphisms of M^* preserve copies. Hence, as expected, the automorphism group of M^* is isomorphic to $G^{\mathbb{N}}$. The action of $G^{\mathbb{N}}$ is defined as follows: if $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ is an element of $G^{\mathbb{N}}$ and (x, i) is in M^*, then

$$\varphi(a, i) = (\varphi_i(a), i),$$

with the convention that for every $g \in G$, $g(\star) = \star$.

Remark 15. It might seem more natural to consider the actual **product structure** of M, whose universe is $M^{\mathbb{N}}$, and where predicates and functions work coordinatewise. There is indeed no problem to define functions here. The automorphism group of the product structure of M is also the product $G^{\mathbb{N}}$. However, the extension property does not carry to the product structure unless it is in some sense uniform[3]. The homogeneity and the existence property do carry, though, and the proofs are similar to those for the juxtaposed structure.

Proposition 16. Let M be a metric structure of diameter smaller than 1. If the structure M is exactly ultrahomogeneous, then so is M^*.

Proof. Let f be an isomorphism between two finite substructures of M^*. Since f preserves the predicates C_n, we can write f as a sequence $(f_n)_{n\in\mathbb{N}}$, where f_n is an isomorphism between finite substructures of M. We apply the ultrahomogeneity of M to each f_n and extend it to an automorphism φ_n of the whole structure M. Then the sequence $(\varphi_n)_{n\in\mathbb{N}}$ is an automorphism of M^* which extends f. □

The following proposition gives a description of types in the juxtaposed structure: they are "products" of types in each copy. To simplify the notation, we only state it for pairs, but it works exactly the same for bigger tuples.

Proposition 17. Let M be a metric structure of diameter smaller than 1 and let a and b be elements of M. Let i and j be two distinct indices in $\mathbb{N} \setminus \{0\}$ and let p be the quantifier-free type of the tuple $((a, i), (b, j))$ in M^*. Let also p_a and p_b be the quantifier-free types of a and b in M respectively. Then the set of realizations of p in M^* is described as follows:

$$p(M^*) = \{((a', i), (b', j)) : a' \in p_a(M) \text{ and } b' \in p_b(M)\}.$$

Proof. Let $\bar{c}$ in M^* have the same quantifier-free type as $((a, i), (b, j))$. Since the predicates C_i and C_j are in the language, we can write $\bar{c}$ as $((a', i), (b', j))$, with a' and b' in M. Let now θ be a formula on $(M^*)^2$. If θ depends on its two variables in M^*, then $\theta(\bar{c}) = 1 = \theta((a, i), (b, j))$. If not, say θ is a formula on $M \times \{i\}$, then it is induced by a formula θ_M on M (the formula θ_M is the projection on M of θ) and we have

$$\theta_M(a') = \theta(a', i) = \theta(\bar{c}) = \theta((a, i), (b, j)) = \theta(a, i) = \theta_M(a),$$

and a' has the same quantifier-free type as a. Similarly, b' has the same quantifier-free type as b.

Conversely, any tuple of the form $((a', i), (b', j))$, where a' and b' have the same quantifier-free type as a and b respectively, has the same quantifier-free type as $((a, i), (b, j))$. □

We now go over the assumptions of theorem 11 to see how they carry to the juxtaposed structure.

[3]The size of the bigger finite set needs to depend only on the size of the smaller one, see definition 18.

3. The extension property

Definition 18. Let M be a metric structure. We say that M has the **extension property** (Sabok and Malicki say that M has **locally finite automorphisms**) if for every finitely generated substructure A of M and every set P of partial automorphisms of A, there exists a finitely generated substructure B of M that contains A such that every partial automorphism in P extends to a global automorphism of B.

Examples 19. The following structures have the extension property.

- Finite ultrahomogeneous structures.
- The random graph (Hrushovski, [H]).
- The Urysohn space (Solecki, [S2]).
- The measure algebra of the standard probability space (Kechris-Rosendal, [KR, page 32], see also Sabok, [S1, lemma 9.1]).
- The (unit ball of the) separable Hilbert space (see Sabok, [S1, lemma 10.2]).

Proposition 20. Let M be a metric structure of diameter smaller than 1 which satisfies the extension property. Then the juxtaposed structure M^* also satisfies the extension property.

Proof. Let A be a finite substructure of M^*. Since A is finite, it only intersects finitely many copies $M \times \{n\}$. We then apply the extension property in each of those copies and take the union of the obtained sets (together with the special element $(\star, 0)$). □

4. The existence property

Definition 21. Let M be a metric structures and let A, B and C be finitely generated substructures of M such that $A \subseteq B \cap C$. We say that B and C are **independent** over A if for all automorphisms $f_B : B \to B$ and $f_C : C \to C$ that stabilize A and coincide on A, the map $f_B \cup f_C$ extends to an automorphism of the substructure generated by B and C.

Definition 22. Let M be a metric structure. We say that M has the **existence property** (Sabok and Malicki say that M has the **extension property**) if for all finitely generated substructures A, B and C such that $A \subseteq B \cap C$, there exists a finitely generated substructure C' of M that is isomorphic to C such that B and C' are independent over A.

Examples 23. Countable structures with the free amalgamation property (see remark **??**) have the existence property. More generally, structures with a stationary independence relation (in the sense of Tent and Ziegler, [TZ]) have the existence property. In particular, the following structures do.

- The pure infinite set.
- The random graph.
- The Urysohn space and sphere.

Non-example 24. Finite structures fail to have the existence property. Indeed, the whole structure is not independent from itself, which is the only substructure isomorphic to it, over the empty set. There is not enough space in the structure to get independence. In particular, this is the case of our favorite non-example: the two-element structure.

Proposition 25. Let M be a metric structure of diameter smaller than 1 which satisfies the existence property. Then the juxtaposed structure M^* also satisfies the existence property.

Proof. Let A, B and C be finite substructures of M^* such that $A \subseteq B \cap C$. Since B and C are finite, they only intersects finitely many copies $M \times \{n\}$. As for the extension property, we then apply the existence property in each of those copies and take the union of the obtained sets (together with the special element $(\star, 0)$). □

5. Isolation

Sabok and Malicki proposed different definitions for the isolation property in theorem 11. However, it is unclear whether Sabok's property carries to the juxtaposed structure. Thus, we will only consider Malicki's isolation conditions.

5.1. **Relevant tuples.** Malicki's theorem only requires isolation for a sufficiently large family of tuples from the structure. Such families he calls *relevant*.

Definition 26. A family R of tuples of M is called **relevant** if for every tuple $\overline{a}$ in M, there exists a tuple $\overline{b}$ in R such that $G_{\overline{b}} \leqslant G_{\overline{a}}$.

Note that any relevant family of tuples naturally induces a relevant family of tuples of its juxtaposed structure.

Proposition 27. Let M be a metric structure of diameter smaller than 1 and let R be a relevant family of tuples of M. Then the family R^* of all those tuples in M^* whose projection to the every copy $M \times \{n\}$ belongs to R is relevant.

Proof. We only prove it for pairs, but the proof works exactly the same for bigger tuples. Let i and j be two distinct indices in $\mathbb{N}$ and let $\overline{c} = ((a,i),(b,j))$ be a tuple in M^*. The stabilizer of this tuple in $G^{\mathbb{N}}$ is

$$(G^{\mathbb{N}})_{\overline{c}} = \{(\varphi_n) \in G^{\mathbb{N}} : \varphi_i \in G_a \text{ and } \varphi_j \in G_b\}.$$

Now, if both i and j are nonzero, let $\overline{a}' = (a'_1,...,a'_m)$ and $\overline{b}' = (b'_1,...,b'_l)$ be tuples in the relevant family R such that $G_{\overline{a}'} \leqslant G_a$ and $G_{\overline{b}'} \leqslant G_b$. Consider the following tuple of R^*: $\overline{c}' = ((a'_1,i),...,(a'_m,i),(b'_1,j),...,(b'_l,j))$. Then we have $(G^{\mathbb{N}})_{\overline{c}'} \leqslant (G^{\mathbb{N}})_{\overline{c}}$.

If one of the indices is zero, say $i=0$, then $G_a = G_\star = G$ so $G^{\mathbb{N}}_{\overline{c}} = G^{\mathbb{N}}_{(b,j)}$. So if $\overline{b}' = (b'_1,...,b'_l)$ is a tuple in the relevant family such that $G_{\overline{b}'} \leqslant G_b$, the tuple $((b'_1,j),...,(b'_l,j))$ of R^* satisfies that $(G^{\mathbb{N}})_{\overline{b}'} \leqslant (G^{\mathbb{N}})_{\overline{b}}$, proving that R^* is relevant. ☐

5.2. **Direct strong isolation.** We first present one of Malicki's version of the isolation property needed for theorem 11. In fact, we simplify the condition slightly by mentioning only the *local orbit* in the following definition.

Definition 28. Let M be a metric structure and let G be the automorphism group of M. Let $\overline{a}$ be a tuple in M and let p be the quantifier-free type of $\overline{a}$. Let ϵ be a positive real. We say that $\overline{a}$ is **directly ϵ-strongly isolated** if there exist

- a sequence $(\overline{a}_k)_{k\in\mathbb{N}}$ of tuples of quantifier-free type p,
- a sequence $(G_k)_{k\in\mathbb{N}}$ of subgroups of G, and
- a sequence $(\delta_k)_{k\in\mathbb{N}}$ of positive reals

such that

- $G_k[\overline{a}_k] \subseteq B(\overline{a},\epsilon)$,
- if $\overline{a}'$ is a tuple of quantifier-free type p in the ball $B(\overline{a}_k,\delta_k)$, then the types $\mathrm{qftp}(\overline{a}'/\overline{a})$ and $\mathrm{qftp}(\overline{a}'/\overline{a}_k)$ are realized in $G_k[\overline{a}_k]$, and
- for every sequence $(g_k)_{k\in\mathbb{N}}$ of automorphisms with $g_k \in G_k$ for all k, there exists an automorphism g in G such that for all k, we have $g \restriction G_k[\overline{a}_k] = g_k \restriction G_k[\overline{a}_k]$.

The last two conditions are conditions of *local relative saturation* and *local relative homogeneity*. DESSIN?

Definition 29. We say that a tuple in M is **directly strongly isolated** if it is directly ϵ-strongly isolated for every positive ϵ.

Example 30. (Malicki) In the Urysohn space, every tuple is directly strongly isolated.

Remark 31. If the structure M is discrete, then the condition of being directly strongly isolated is empty. Indeed, if $\bar{a}$ is any tuple in M and ϵ is any positive real, then $\bar{a}$ is directly ϵ-strongly isolated by the constant sequences $(\bar{a})_{k\in\mathbb{N}}$ and $(\{id_M\})_{k\in\mathbb{N}}$, with any sequence $(\delta_k)_{k\in\mathbb{N}}$ of positive reals. In particular, every tuple in the two-element structure is directly strongly isolated.

We are now ready to state Malicki's theorem in a precise way.

Theorem 32. (Malicki) Let M be an exactly ultrahomogeneous metric structure. Let R be a relevant family of tuples of M. Assume that M has the extension property and the existence property, and that every tuple in R is directly strongly isolated. Then the automorphism group of M is Steinhaus and thus satisfies the automatic continuity property.

Remark 33. If the structure M is discrete, the theorem more or less restates the result that a Fraïssé structure with both the extension property and the free amalgamation property has the automatic continuity property.

We prove that direct strong isolation carries to the juxtaposed structure.

Proposition 34. Let M be a metric structure of diameter smaller than 1. Let ϵ be a positive real. Let a and b be two directly ϵ-strongly isolated elements of M. Let i and j be two distinct indices in $\mathbb{N}$. Then the tuple $\bar{c} = ((a,i),(b,j))$ is also directly ϵ-strongly isolated in M^*.

Proof. Let (a_k), (G_k), (δ_k) and (b_k), (H_k), (ϵ_k) witness the direct ϵ-strong isolation of a and b respectively. We prove that the tuple $\bar{c}$ is then directly ϵ-strongly isolated by the sequences $(\bar{c}_k)$, (K_k) and (η_k), where

- $\bar{c}_k = ((a_k,i),(b_k,j))$,
- $K_k = \{(\varphi_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}} : \varphi_i \in G_k \text{ and } \varphi_j \in H_k\}$ and
- $\eta_k = \min(\delta_k, \epsilon_k)$.

First note that K_k is indeed a subgroup of $G^{\mathbb{N}}$. Besides, any element of $K_k[\bar{c}_k]$ is of the form $((g_k(a_k),i),(h_k(b_k),j))$, with $g_k \in G_k$ and $h_k \in H_k$. The isolation of a and b gives that $g_k(a_k) \in B(a,\epsilon)$ and $h_k(b_k) \in B(b,\epsilon)$, and since we take the supremum distance on tuples, we have that $K_k[c_k]$ is contained in the ball $B(\bar{c},\epsilon)$.

By proposition 17, all the tuples $\bar{c}_k$ have the same quantifier-free type as $\bar{c}$. Let now $\bar{c}'$ be a tuple in the ball $B(\bar{c}_k,\eta_k)$ that has the same quantifier-free type as $\bar{c}$. We can write it $\bar{c}' = ((a',i),(b',i))$ and, by proposition 17 again, the elements a' and b' have the same quantifier-free type as a and b respectively. We can thus find realizations a_1, a_2 in $G_k[a_k]$ and b_1, b_2 in $H_k[b_k]$ of $\mathrm{qftp}(a'/a)$, $\mathrm{qftp}(a'/a_k)$ and $\mathrm{qftp}(b'/a)$, $\mathrm{qftp}(b'/b_k)$. Now the tuples $\bar{c}_1 = ((a_1,i),(b_1,j))$ and $\bar{c}_2 = ((a_2,i),(b_2,j))$ are realizations of $\mathrm{qftp}(\bar{c}'/\bar{c})$ and $\mathrm{qftp}(\bar{c}'/\bar{c}_k)$ in $K_k[\bar{c}_k]$ (we use proposition 17 once again).

Finally, let $(\varphi^k)_{k\in\mathbb{N}}$ be a sequence of automorphisms of M^*, with $\varphi^k \in K_k$ for all k. We can write each φ^k as a sequence $(\varphi^k_n)_{n\in\mathbb{N}}$, with $\varphi^k_i \in G_k$ and $\varphi^k_j \in H_k$. We apply the local relative homogeneity conditions to the sequences $(\varphi^k_i)_{k\in\mathbb{N}}$ and $(\varphi^k_j)_{k\in\mathbb{N}}$ to get automorphisms φ_i and φ_j of M such that for all k, we have $\varphi_i \restriction G_k[a_k] = \varphi^k_i \restriction G_k[a_k]$ and $\varphi_j \restriction H_k[b_k] = \varphi^k_j \restriction H_k[a_k]$. Then the automorphism $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ of M^* defined by $\varphi_n = \varphi_i$

$$\varphi_n = \begin{cases} \varphi_i & \text{if } n = i \\ \varphi_j & \text{if } n = j \\ \mathrm{id}_M & \text{otherwise} \end{cases}$$

satisfies that $\varphi \restriction K_k[\bar{c}_k] = \varphi_k \restriction K_k[\bar{c}_k]$, which completes the proof. □

The proof readily adapts to bigger tuples. As a consequence, we obtain that the isolation condition in theorem 32 carries to the product.

Corollary 35. Let M be a metric structure of diameter smaller than 1. Let R be a relevant family of tuples of M and let R^* be the relevant family of tuples of M^* in proposition 27. If every tuple in R is directly strongly isolated, then so is every tuple in R^*.

This finally yields that this better set of reasons to have a Steinhaus automorphism group carries to the juxtaposed structure.

Theorem 36. Let M be an exactly ultrahomogeneous metric structure of diameter smaller than 1 and let G be its automorphism group. Let R be a relevant family of tuples of M. Assume that M has the extension property and the existence property, and that every tuple in R is directly strongly isolated. Then the group $G^{\mathbb{N}}$ is Steinhaus and thus satisfies the automatic continuity property.

Corollary 37. The group $\mathrm{Iso}(\mathbb{U}_1)^{\mathbb{N}}$ satisfies the automatic continuity property.

Remark 38. The metric on the Urysohn space is not bounded. However, it is equivalent to the following metric

$$d'(x,y) = \frac{d(x,y)}{1+d(x,y)},$$

which is bounded by 1. Moreover, the isometry groups of $(\mathbb{U}, d)$ and $(\mathbb{U}, d')$ are the same. Thus, we can apply the previous results to the Urysohn space too: we also get the automatic continuity property for $\mathrm{Iso}(\mathbb{U})^{\mathbb{N}}$.

5.3. **Indirect strong isolation.**

Definition 39. Let M be a metric structure and let G be the automorphism group of M. Let $\bar{a}$ be a tuple in M and let p be the quantifier-free type of $\bar{a}$. Let ϵ be a positive real. We say that $\bar{a}$ is **indirectly ϵ-strongly isolated** if there exist

- a sequence $(\bar{a}_k)_{k\in\mathbb{N}}$ of tuples of quantifier-free type p,
- a sequence $(G_k)_{k\in\mathbb{N}}$ of subgroups of G,
- a sequence $(\delta_k)_{k\in\mathbb{N}}$ of positive reals, and
- a sequence $(X_k)_{k\in\mathbb{N}}$ of metric substructures of M

such that

- X_k is exactly homogeneous and satisfies the extension property and the existence property,
- X_k is invariant under the action of G_l, for all l in $\mathbb{N}$,
- $G_k \restriction X_k = \mathrm{Aut}(X_k)$ and every element of $\mathrm{Aut}(X_k)$ extends uniquely to an element of G_k,
- $G_k[\bar{a}_k] \subseteq B(\bar{a}, \epsilon)$,
- if $\overline{a}'$ is a tuple of quantifier-free type p in the ball $B(\overline{a}_k, \delta_k)$, then the types $\mathrm{qftp}(\overline{a}'/\overline{a})$ and $\mathrm{qftp}(\overline{a}'/\overline{a}_k)$ are realized in $G_k[\overline{a}_k]$, and
- for every sequence $(g_k)_{k\in\mathbb{N}}$ of automorphisms with $g_k \in G_k$ for all k, there exists an automorphism g in G such that for all k, we have $g \restriction X_k = g_k \restriction X_k$.

Remark 40. In the definition of direct isolation, the role of X_k in the local relative homogeneity condition is played by the local orbit $G_k[\bar{a}_k]$, although the local orbit is not necessarily a substructure of M (let alone an ultrahomogeneous substructure with the extension property and the existence property).

Definition 41. We say that a tuple in M is **indirectly strongly isolated** if it is indirectly ϵ-strongly isolated for every positive ϵ.

Examples 42. (Malicki)

- In the measure algebra of a standard probability space X, every partition of X into positive measure sets is indirectly strongly isolated.

- In the Hilbert space, every orthonormal tuple is indirectly strongly isolated.

Here is the indirect version of Malicki's result.

Theorem 43. (Malicki) Let M be an exactly ultrahomogeneous metric structure. Let R be a relevant family of tuples of M. Assume that M has the extension property and the existence property, and that every tuple in R is indirectly strongly isolated. Then the automorphism group of M is Steinhaus and thus satisfies the automatic continuity property.

We now prove that indirect strong isolation carries to the juxtaposed structure.

Proposition 44. Let M be a metric structure of diameter smaller than 1. Let ϵ be a positive real. Let a and b be two indirectly ϵ-strongly isolated elements of M. Let i and j be two distinct indices in $\mathbb{N}$. Then the tuple $\bar{c} = ((a,i),(b,j))$ is also indirectly ϵ-strongly isolated in M^*.

Proof. Let (a_k), (G_k), (δ_k), (X_k) and (b_k), (H_k), (ϵ_k), (Y_k) witness the indirect ϵ-strong isolation of a and b respectively. We prove that the tuple $\bar{c}$ is then indirectly ϵ-strongly isolated by the sequences $(\bar{c}_k)$, (K_k), (η_k) and (Z_k), where

- $\bar{c}_k = ((a_k,i),(b_k,j))$,
- $K_k = \{(\varphi_n)_{n\in\mathbb{N}} \in G^{\mathbb{N}} : \varphi_i \in G_k \text{ and } \varphi_j \in H_k\}$,
- $\eta_k = \min(\delta_k, \epsilon_k)$, and
- $Z_k = (X_k \times \{i\}) \cup (Y_k \times \{j\})$.

As in proposition 34, we have that K_k is a subgroup of $G^{\mathbb{N}}$, that $K_k[\bar{c}_k]$ is contained in $B(\bar{c},\epsilon)$, that the tuples $\bar{c}_k$ have the same quantifier-free type as $\bar{c}$ and the local relative saturation property.

Note that Z_k is indeed a substructure of M^*, whose automorphism group is $\mathrm{Aut}(X_k)\times\mathrm{Aut}(Y_k)$.

The proofs of propositions 16, 20 and 25 show that the structure Z_k is exactly ultrahomogeneous and satisfies the extension property and the existence property.

Besides, the restriction of K_k to Z_k is $(G_k \upharpoonright X_k) \times (H_k \upharpoonright Y_k)$, which coincides with $\mathrm{Aut}(X_k) \times \mathrm{Aut}(Y_k) = \mathrm{Aut}(Z_k)$.

Moreover, $K_l(Z_k) = (G_l[X_k] \times \{i\}) \cup (H_l[Y_k] \times \{j\})$. Thus, since X_k and Y_k are invariant under the actions of G_l and H_l respectively, the structure Z_k is invariant under the action of K_l.

Finally, let $(\varphi^k)_{k\in\mathbb{N}}$ be a sequence of automorphisms of M^*, with $\varphi^k \in K_k$ for all k. We can write each φ^k as a sequence $(\varphi^k_n)_{n\in\mathbb{N}}$, with $\varphi^k_i \in G_k$ and $\varphi^k_j \in H_k$. We apply the local relative homogeneity conditions to the sequences $(\varphi^k_i)_{k\in\mathbb{N}}$ and $(\varphi^k_j)_{k\in\mathbb{N}}$ to get automorphisms φ_i and φ_j of M such that for all k, we have $\varphi_i \upharpoonright X_k = \varphi^k_i \upharpoonright X_k$ and $\varphi_j \upharpoonright Y_k = \varphi^k_j \upharpoonright Y_k$. Then the automorphism $\varphi = (\varphi_n)_{n\in\mathbb{N}}$ of M^* defined by $\varphi_n = \varphi_i$

$$\varphi_n = \begin{cases} \varphi_i & \text{if } n = i \\ \varphi_j & \text{if } n = j \\ \mathrm{id}_M & \text{otherwise} \end{cases}$$

satisfies that $\varphi \upharpoonright Z_k = \varphi_k \upharpoonright Z_k$, which completes the proof. □

The proof readily adapts to bigger tuples. As a consequence, we obtain that the isolation condition in theorem 43 carries to the product.

Corollary 45. Let M be a metric structure of diameter smaller than 1. Let R be a relevant family of tuples of M and let R^* be the relevant family of tuples of M^* in proposition 27. If every tuple in R is indirectly strongly isolated, then so is every tuple in R^*.

This finally yields that this indirect better set of reasons to have a Steinhaus automorphism group also carries to the juxtaposed structure.

Theorem 46. Let M be an exactly ultrahomogeneous metric structure of diameter smaller than 1 and let G be its automorphism group. Let R be a relevant family of tuples of M. Assume that M has the extension property and the existence property, and that every tuple in R is indirectly strongly isolated. Then the group $G^{\mathbb{N}}$ is Steinhaus and thus satisfies the automatic continuity property.

Corollary 47. The groups $\mathrm{Aut}(\mu)^{\mathbb{N}}$ and $\mathcal{U}(\ell^2)^{\mathbb{N}}$ satisfy the automatic continuity property.

6. Connected groups with ample generics

Recall that when G is a Polish group that has ample generics, then $G^{\mathbb{N}}$ also has ample generics (proposition 10). With Le Maître, we noticed that this could be generalized to the group $L^0(X,\mu;G)$ of G-valued random variables on a standard probability space[4] (X,μ). Indeed, the group $G^{\mathbb{N}}$ can be viewed as that of G-valued random variables on $\mathbb{N}$. That is what lead us to an answer to Kechris and Rosendal's question.

Equip the group $L^0(X,\mu;G)$ with the topology of convergence in measure. More concretely, if d is a compatible metric on G, then the topology on $L^0(X,\mu;G)$ is induced by the metric

$$d^R(g,h)=\int_X d(g(x),h(x))d\mu(x).$$

Moreover, whenever G is a Polish group, $L^0(X,\mu;G)$ is connected (it is even contractible, see [K1, proposition 19.7]), hence cannot be a topological subgroup of the totally disconnected group S_∞. Together with the following theorem, this yields a family of examples of connected Polish groups with ample generics, answering Kechris and Rosendal's question.

Theorem 48. (with Le Maître) Let G be a Polish group with ample generics. Then the group $L^0(X,\mu;G)$ also ample generics.

Proof. We wish to prove that for every n in $\mathbb{N}$, the diagonal action of $L^0(X,\mu;G)$ on $L^0(X,\mu;G)^n$ admits a comeager orbit. Here too, there is a natural identification of $L^0(X,\mu;G)^n$ with $L^0(X,\mu;G^n)$.

Let $\bar\varphi$ be an element of G^n whose orbit is comeager and consider the constant function $\bar f : x\mapsto\bar\varphi$ in $L^0(X,\mu;G^n)$. We show that the orbit of $\bar f$ in $L^0(X,\mu;G^n)$ is comeager.

First, let us remark that the orbit of $\bar f$ is thus described:

$$L^0(X,\mu;G)\cdot\bar f=\{\bar g\in L^0(X,\mu;G^n):\bar g(x)\in G\cdot\bar\varphi\text{ for }\mu\text{-almost every }x\}.$$

Indeed, if $\bar g$ is in the orbit of $\bar f$, then $\bar g$ is clearly in the set above. Conversely, assume that $\bar g(x)$ is in $G\cdot\bar\varphi$ almost everywhere. There is a Borel subset B of X with measure 1 such that for every x in B, there exists an element h_x in G such that $\bar g(x)=h_x\cdot\bar\varphi$. We would like to find those h_x in a measurable way. For this, we apply the Jankov-von Neumann uniformization theorem to the following Borel set

$$S=\{(x,h_x)\in X\times G:[x\in B\text{ and }\bar g(x)=h_x\cdot\bar\varphi]\text{ or }x\notin B\},$$

which projects to the whole space X. We thus obtain a map h in $L^0(X,\mu;G)$ whose graph is contained in S, that is, $\bar g=h\cdot\bar f$, hence $\bar g$ belongs to orbit of $\bar f$.

Now the orbit of $\bar\varphi$ is comeager in G^n so it contains a countable intersection $\bigcap_{k\in\mathbb{N}}U_k$ of dense open subsets of G^n. For every k in $\mathbb{N}$, consider the set

$$V_k=\{\bar g\in L^0(X,\mu;G^n):\bar g(x)\in U_k\text{ for }\mu\text{-almost every }x\}.$$

The intersection of all V_k's is contained in the orbit of $\bar f$. Therefore, it remains to prove that the V_k's are dense and G_δ.

[4] Say, the unit interval together with its Lebesgue measure.

To see that V_k is dense, let $\bar{h}$ be any measurable map from X to G^n and let ϵ be a positive real. We apply the Jankov-von Neumann theorem to the Borel set

$$\{(x, \bar{g}_x) \in X \times G^n : d_L(\bar{g}_x, \bar{h}(x)) < \epsilon \text{ and } \bar{g}_x \in U_k\},$$

which projects to the whole space X. Thereby, we obtain an element $\bar{g}$ of V_k such that $d^R(\bar{g}, \bar{h}) < \epsilon$.

Finally, V_k can be written as the intersection $\bigcap_{m\in\mathbb{N}} V_{k,m}$, where

$$V_{k,m} = \left\{\bar{g} \in L^0(X, \mu; G^n) : \mu\left(\{x \in X : \bar{g}(x) \in U_k\}\right) > 1 - 2^{-m}\right\}.$$

Since the topology on $L^0(X; \mu; G^n)$ is given by convergence in measure, each of the sets $V_{k,m}$ is open in $L^0(X, \mu; G^n)$, proving that V_k is G_δ, which completes the proof. □

As a consequence, we notably obtain that every Polish group with ample generics embeds in a connected (even contractible) one with ample generics.

Remark 49. We have another example of a connected Polish group with ample generics: the full group of a quasi-measure-preserving hyperfinite equivalence relation (see [KLM]). It is interesting to note that it is also a subgroup of $L^0(X, \mu; S_\infty)$.

Thus, the question remains open of whether Polish groups with ample generics exist that are not subgroups of $L^0(X, \mu; S_\infty)$.

7. Concluding remarks

With theorem 48 at hand, it is natural to ask the same question of the group $L^0(X, \mu; G)$. However, in this case, the construction of the juxtaposed structure would not make much sense. Rather, the group $L_0([0, 1], G)$ is the automorphism group of a randomization[5] of the structure $\mathbf{M}$, which is the counterpart of the product structure of $\mathbf{M}$. This randomization remains exactly ultrahomogeneous if the original is. As in the proof of theorem 48, in order to carry properties from the structure to its randomization, our main tool is the Jankov-van Neumann theorem. But again, we need some amount of uniformity to apply it.

Answering this question of Kechris and Rosendal came as a very pleasant surprise. Apart from that, though, the interest of our results lies essentially in that we uncover the automatic continuity property for new groups. However, this does not shed any new light on Sabok's and Malicki's conditions. It would be nice to know if and how the isolation properties can be simplified, and how much of them is really needed for automatic continuity.

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[5]Tomás insisted that this structure does not deserve the name of randomization; I apologize to him...

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